International Journal of Heat and Mass Transfer 52 (2009) 1635–1642

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

A two-stage LGSM to identify time-dependent heat source through an internal measurement of temperature

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ARTICLE INFO

Article history:
Received 12 May 2008
Received in revised form 21 August 2008
Available online 27 November 2008

Keywords:
Inverse problem
Two-stage Lie-group shooting method
Time-dependent heat source

ABSTRACT

We consider an inverse problem for estimating an unknown time-dependent heat source $H(t)$ in a heat conduction equation $T_t(x,t) = T_{xx}(x,t) + H(t)$, with the aid of an extra measurement of temperature at an internal point. The Lie-group shooting method (LGSM) was used in the solution of this inverse problem; however, when the data are acquired at an internal point we require to develop a two-stage Lie-group shooting method (TSLGSM) to solve it. This novel approach is examined through numerical examples to convince that it is a rather accurate and efficient method, whose estimation error is small even for the identification of discontinuous and oscillatory heat source under large noise.

1. Introduction

In the parabolic type diffusion problems the source terms are usually not easy to detect directly. In practice, there are many researches on the inverse source identification problem to determine the source terms since 1970s. This study aims to estimate as accurately as possible the time-varying heat source by solving an inverse heat conduction problem under an overspecified internal data. The estimation is based on a transient temperature measurement undertaken by a thermocouple on an internal point of a heat conducting rod.

Applications of inverse methods span over many heat transfer related topics. Sometimes the temperature and heat flux data on the boundary are known and one wants to determine the material properties. Those problems are often referred to as parameter identification problems in the literature [1,2]. Most inverse problems belong to a family of problems that have an inherited ill-posed property. Since the interest in these methods begun with one of the first published paper by Stolz [3] in the 1960, the applications nowadays range over many scientific fields. Those fields include solid mechanics, fluid dynamics and heat transfer, to name only a few.

The parameter determination in partial differential equations from overspecified data play a crucial role in applied mathematics and physics. These problems are widely encountered in the modeling of physical phenomena [4–7]. Here, we consider an inverse problem of finding an unknown heat source $H(t)$ in a one-dimensional heat conduction equation, of which one needs to find the temperature distribution $T(x,t)$ as well as the heat source $H(t)$ that simultaneously satisfy

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial T(x,t)}{\partial t} - H(t), \quad 0 < x < \ell, 0 < t \leqslant t_f, \tag{1}$$

$$T(0,t) = F_0(t), \quad T(\ell,t) = F_\ell(t), \tag{2}$$

$$T(x,0) = f(x). \tag{3}$$

Because the above problem has an unknown function $H(t)$, it cannot be solved directly. In the above, ℓ is a length of the heat conducting rod, and t_f is a terminal time.

A new method will be developed to estimate the unknown heat source $H(t)$ of the above inverse problem, which is subjected to the above boundary conditions and initial condition, as well as an overspecified temperature measurement at an internal point x_m:

$$T(x_m,t) = F_m(t). \tag{4}$$

For the inverse problem governed by Eqs. (1)–(4) there are many studies as can be seen from the papers by Cannon and Duchateau [8] for identifying $H(u)$, and Savateev [9] and Borukhov and Vabishchevich [10] for identifying $H(x,t)$ with additive or seperable space and time. Many researchers sought the heat source as a function of only space or time, for example, Farcas and Lesnic [11], Ling et al. [12], and Yan et al. [13].

The model problem presented here used to describe a heat transfer process with a time-dependent source produces the temperature at a given point x_m in the spatial domain at time t. Thus, the purpose of solving this inverse problem can be viewed as an inverse control problem to identify the source control parameter that produces at any given time a desired temperature at a given point x_m in the spatial domain. The traditional approach in solving problems of this sort approximately consists in reduction to the first

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doi:10.1016/j.ijheatmasstransfer.2008.09.021

Nomenclature

$\mathbf{A}$	augmented matrix
a, b	coefficients defined in Eqs. (29), (32), (38), (50)
$\mathbf{f}$	$2n$-dimensional vector field in Eq. (14)
$\hat{\mathbf{f}}$	$:= \mathbf{f}(\hat{x}, \hat{\mathbf{y}})$
$f(x)$	initial temperature function
$F_0(t)$	left-boundary temperature function
$F_\ell(t)$	right-boundary temperature function
$F_{\rm m}(t)$	temperature function at $x_{\rm m}$
$\mathbf{F}$	$:= \hat{\mathbf{f}}/\|\hat{\mathbf{y}}\|$
$\mathbf{F}_1$	the first n components of $\mathbf{F}$
$\mathbf{F}_2$	the last n components of $\mathbf{F}$
$\hat{F}(t_i)$	$:= rF_0(t_i) + (1-r)F_{\rm m}(t_i)$
$\hat{F}_{\rm m}(t_i)$	$:= F_{\rm m}(t_i) + sR(i)$
$\mathbf{g}$	$2n+1$-dimensional Minkowski metric
$\mathbf{G}$	an element of Lorentz group
$\mathbf{G}_i, i = 1, \ldots, K$	elements of Lorentz group
$\mathbf{G}(r)$	an element of Lorentz group
$\mathbf{G}(x_{\rm m})$	an element of Lorentz group
G_0^0	the 00-th component of $\mathbf{G}$
$H(t)$	time-dependent heat source
H_i	$:= H(t_i)$
$\mathbf{h}$	right-hand sides of Eqs. (8) and (9)
$\hat{\mathbf{h}}_1$	$:= \mathbf{h}((1-r)x_{\rm m}, \hat{\mathbf{T}}_1)$
$\hat{\mathbf{h}}_2$	$:= \mathbf{h}(rx_{\rm m} + (1-r)\ell, \hat{\mathbf{T}}_2)$
$\hat{h}_1^i$	the i-th component of $\hat{\mathbf{h}}_1$
$\mathbf{I}_{2n}$	$2n$-dimensional unit matrix
ℓ	length of rod
$\|\bullet\|$	Euclidean norm
$\mathbb{M}^{2n+1}$	$2n+1$-dimensional Minkowski space
n	number of discretized time points
r	weighting factor
$R(i)$	random numbers
s	level of noise
$SO_{\rm o}(2n, 1)$	$2n+1$-dimensional Lorentz group
$so(2n, 1)$	the Lie algebra of $SO_{\rm o}(2n, 1)$
S	temperature gradient
$\mathbf{S}$	temperature gradient vector of S^i
$\mathbf{S}^0$	temperature gradient vector at $x = 0$
$\mathbf{S}^\ell$	temperature gradient vector at $x = \ell$
$\mathbf{S}^{\rm m}$	temperature gradient vector at $x = x_{\rm m}$
$\hat{\mathbf{S}}_1$	$:= r\mathbf{S}^0 + (1-r)\mathbf{S}^{\rm m}$
$\hat{\mathbf{S}}_2$	$:= r\mathbf{S}^{\rm m} + (1-r)\mathbf{S}^\ell$
$S^i(x)$	$:= S(x, t_i)$
t	time
$t_{\rm f}$	final time
t_i	$:= i\Delta t$
Δt	time stepsize
T	temperature
$\mathbf{T}$	temperature vector of T^i
$\mathbf{T}^0$	temperature vector at $x = 0$
$\mathbf{T}^\ell$	temperature vector at $x = \ell$
$\mathbf{T}^{\rm m}$	temperature vector at $x = x_{\rm m}$
$\hat{\mathbf{T}}_1$	$:= r\mathbf{T}^0 + (1-r)\mathbf{T}^{\rm m}$
$\hat{\mathbf{T}}_2$	$:= r\mathbf{T}^{\rm m} + (1-r)\mathbf{T}^\ell$
$T^i(x)$	$:= T(x, t_i)$
$\hat{T}_1^i$	the i-th component of $\hat{\mathbf{T}}_1$
$\hat{T}_2^i$	the i-th component of $\hat{\mathbf{T}}_2$
x	space variable
Δx	mesh size of x
$x_{\rm m}$	temperature measuring point
x_y	$:= x_{\rm m}\|\mathbf{y}^{\rm m} - \mathbf{y}^0\|$
$\hat{x}$	$:= (1-r)x_{\rm m}$
$\mathbf{X}$	$2n+1$-dimensional augmented vector
$\mathbf{X}_k$	numerical value of $\mathbf{X}$ at the k-th spatial step
$\mathbf{X}^0$	the value of $\mathbf{X}$ at $x = 0$
$\mathbf{X}^{\rm m}$	the value of $\mathbf{X}$ at $x = x_{\rm m}$
$\mathbf{y}$	$2n$-dimensional vector defined in Eq. (14)
$\mathbf{y}^0$	the value of $\mathbf{y}$ at $x = 0$
$\mathbf{y}^{\rm m}$	the value of $\mathbf{y}$ at $x = x_{\rm m}$
$\mathbf{y}^\ell$	the value of $\mathbf{y}$ at $x = \ell$
$\|\hat{\mathbf{y}}_1\|$	$:= \sqrt{\|\hat{\mathbf{T}}_1\|^2 + \|\hat{\mathbf{S}}_1\|^2}$
$\|\hat{\mathbf{y}}_2\|$	$:= \sqrt{\|\hat{\mathbf{T}}_2\|^2 + \|\hat{\mathbf{S}}_2\|^2}$
Z	$:= \exp(x_y/\eta)$
Z_1	$:= \exp(x_{\rm m}\|\mathbf{y}^{\rm m} - \mathbf{y}^0\|/\eta_1)$
Z_2	$:= \exp((\ell - x_{\rm m})\|\mathbf{y}^\ell - \mathbf{y}^{\rm m}\|/\eta_2)$

Greek symbols

ϵ	converegnce criterion
η	coefficient defined in Eqs. (35) and (47)
η_1	coefficient defined in Eq. (64)
η_2	coefficient defined in Eq. (72)
θ	intersection angle of $\mathbf{y}^{\rm m} - \mathbf{y}^0$ and $\mathbf{y}^0$
θ_1	intersection angle of $\mathbf{y}^{\rm m} - \mathbf{y}^0$ and $\mathbf{y}^0$
θ_2	intersection angle of $\mathbf{y}^\ell - \mathbf{y}^{\rm m}$ and $\mathbf{y}^{\rm m}$

Subscripts and superscripts

i	index
K	index
t	transpose

kind Volterra integral equation, and then some regularization techniques are used to solve the ill-posed problem. According to this type formulation, Maalek Ghaini [14] has proven the existence, uniqueness and stability problems; however, no numerical procedures and examples were presented. More interestingly, Yan et al. [13] have transformed the above problem into a three-point boundary value problem.

The heat source identification of $H(t)$ is one of the inverse problems for the applications in heat conduction engineering by detecting the thermal source. The inverse problems are those in which one would like to determine the causes for an observed effect. One of the characterizing properties of many of the inverse problems is that they are usually ill-posed, in the sense that a solution that depends continuously on the data do not exist. For this inverse problem of heat source identification the observed effect is the temperature measurement $T(x_{\rm m}, t)$ at an internal point $x = x_{\rm m}$ on the rod. We are interesting to search the cause of the unknown heat source $H(t)$ in Eq. (1), which induces the effect we observe through measurement. For the inverse problems the measurement error may often lead to a large discrepancy from the true cause.

Our approach of the above inverse problem is by using a semi-discretization together with the group-preserving scheme (GPS) developed previously by Liu [15] for ordinary differential equations (ODEs). Recently, Liu [16–18] has extended the GPS technique to solve the boundary value problems (BVPs), and the numerical results reveal that the Lie-group method is a rather promising technique to effectively calculate the two-point BVPs. In the construction of the Lie-group method for the calculations of BVPs, Liu [16] has introduced the idea of one-step GPS by utilizing the closure property of Lie-group, and hence, the new shooting method has been named the Lie-group shooting method (LGSM). Chang et al. [19] have employed the LGSM to solve a backward heat conduction problem with a high performance. Liu [7] has employed the LGSM technique to solve accurately the inverse heat conduction problems of identifying nonhomogeneous heat conductivity functions.

Recently, Liu [20] has developed a two-stage Lie-group shooting method (TSLGSM) for three-point boundary value problems of second-order ordinary differential equations. The above paper is greatly extended the capability of shooting technique on the solutions of BVPs. Developing here is a new TSLGSM for the inverse problem of heat source identification governed by Eqs. (1)–(4).

It is interesting to note that the new method of TSLGSM does not require any a priori regularization when applying it to the inverse problem of heat source identification, and also exhibits several advantages than other methods. It would be clear that the new method can greatly reduce the computational time and is very easy to implement on the calculations of inverse problem of heat source identification. Especially, the present method of TSLGSM would provide much better computational results than others, which in turns greatly suggest us to use the TSLGSM in the calculations of this inverse problems.

2. Mathematical backgrounds

We will develop a numerical method to estimate the heat source $H(t)$ based on the numerical method of line, which leads to a set of ODEs. In order to explore our new method in self-content, let us first briefly sketch the group-preserving scheme (GPS) for ODEs and one-step GPS for the Lie-group in this section. The readers may refer the author's papers listed in the References for a detailed treatment.

2.1. A semi-discretization

As that done by Chang et al. [21], Eq. (1) is transformed into the following equations:

$$\frac{\partial T(x,t)}{\partial x} = S(x,t), \tag{5}$$

$$\frac{\partial S(x,t)}{\partial x} = \frac{\partial T(x,t)}{\partial t} - H(t). \tag{6}$$

Then, by using a semi-discretization method to discretize the quantities of $T(x,t)$ and $S(x,t)$ in the time domain, we can obtain a system of ODEs for T and S with x as an independent variable. The Lie-group method as first developed by Liu [22] for the parameter estimation is extended and applied to the following discretized equations:

$$\frac{\partial T^i(x)}{\partial x} = S^i(x), \quad i = 1,\dots,n, \tag{7}$$

$$\frac{\partial S^i(x)}{\partial x} = \frac{T^{i+1}(x) - T^{i-1}(x)}{2\Delta t} - H_i, \quad i = 1,\dots,n-1, \tag{8}$$

$$\frac{\partial S^n(x)}{\partial x} = \frac{3T^n(x) - 4T^{n-1}(x) + T^{n-2}(x)}{2\Delta t} - H_n, \tag{9}$$

where $\Delta t = t_f/n$ is a uniform time increment, and $t_i = i\Delta t$ are the discretized times of which the measurement is sampling by a rate Δt. $T^i(x) = T(x,t_i)$, $S^i(x) = S(x,t_i)$ and $H_i = H(t_i)$ are the discretized quantities at the nodal points of time.

When $i = 1$, the term $T^0(x)$ appeared in Eq. (8) is determined by the initial condition (3). While the central difference is used in Eq. (8), we may use the backward difference in Eq. (9) at the last time point in order to maintain the same second-order accuracy.

The three known boundary conditions are given by

$$T^i(0) = F_0(t_i), \qquad i = 1,\dots,n, \tag{10}$$

$$T^i(x_m) = F_m(t_i), \quad i = 1,\dots,n, \tag{11}$$

$$T^i(\ell) = F_\ell(t_i), \qquad i = 1,\dots,n, \tag{12}$$

which are obtained from Eqs. (2) and (4) by discretizations.

2.2. The GPS

Let us write Eqs. (7)–(9) as in a vector form:

$$\mathbf{y}' = \mathbf{f}(x,\mathbf{y}), \tag{13}$$

where the prime denotes the differential with respect to x, and

$$\mathbf{y} := \begin{bmatrix} \mathbf{T} \\ \mathbf{S} \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} \mathbf{S} \\ \mathbf{h}(x,\mathbf{T}) \end{bmatrix}, \tag{14}$$

in which $\mathbf{T} = (T^1,\dots,T^n)^t$ and $\mathbf{S} = (S^1,\dots,S^n)^t$. The components of $\mathbf{h}$ represent the right-hand sides of Eqs. (8) and (9). The dependence of $\mathbf{h}$ on x is due to the dependence of initial condition (3) on x.

When both the vector $\mathbf{y}$ and its magnitude $\|\mathbf{y}\| := \sqrt{\mathbf{y}^t\mathbf{y}} = \sqrt{\mathbf{y}\cdot\mathbf{y}}$ were combined into a single augmented vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{bmatrix}, \tag{15}$$

Liu [15] has transformed Eq. (13) into an augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} := \begin{bmatrix} \mathbf{0}_{2n\times 2n} & \frac{\mathbf{f}(x,\mathbf{y})}{\|\mathbf{y}\|} \\ \frac{\mathbf{f}^t(x,\mathbf{y})}{\|\mathbf{y}\|} & 0 \end{bmatrix}\mathbf{X}, \tag{16}$$

where $\mathbf{A}$ is an element of the Lie algebra $so(2n,1)$ satisfying

$$\mathbf{A}^t\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{17}$$

and

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_{2n} & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & -1 \end{bmatrix} \tag{18}$$

is a Minkowski metric. Here, $\mathbf{I}_{2n}$ is the identity matrix, and the superscript t stands for the transpose.

The augmented variable $\mathbf{X}$ can be viewed as a point in the Minkowski space $\mathbb{M}^{2n+1}$, satisfying the cone condition:

$$\mathbf{X}^t\mathbf{g}\mathbf{X} = \mathbf{y}\cdot\mathbf{y} - \|\mathbf{y}\|^2 = 0. \tag{19}$$

Accordingly, Liu [15] has developed a group-preserving scheme (GPS) to guarantee that each $\mathbf{X}_k$ locates on the cone:

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k, \tag{20}$$

where $\mathbf{X}_k$ denotes the numerical value of $\mathbf{X}$ at the discrete x_k, and $\mathbf{G}(k) \in SO_o(2n,1)$ satisfies

$$\mathbf{G}^t\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{21}$$

$$\det\mathbf{G} = 1, \tag{22}$$

$$G_0^0 > 0, \tag{23}$$

where G_0^0 is the 00-th component of $\mathbf{G}$.

2.3. One-step Lie-group transformation

Throughout this paper we use the superscripted symbols $\mathbf{y}^0$ to denote the value of $\mathbf{y}$ at $x = 0$, $\mathbf{y}^m$ to denote the value of $\mathbf{y}$ at $x = x_m$, and $\mathbf{y}^\ell$ the value of $\mathbf{y}$ at $x = \ell$. We first consider the Lie-group shooting method in the interval of $x \in [0, x_m]$.

By sequentially applying scheme (20) on Eq. (16) with a specified left-boundary condition $\mathbf{X}(0) = \mathbf{X}^0$ we can compute the solution $\mathbf{X}(x)$ by the GPS. Assuming that the spatial stepsize used in the GPS is $\Delta x = x_m/K$, and starting from an augmented left-boundary condition $\mathbf{X}^0 = ((\mathbf{y}^0)^t, \|\mathbf{y}^0\|)^t \neq \mathbf{0}$ we will calculate the value $\mathbf{X}^m = ((\mathbf{y}^m)^t, \|\mathbf{y}^m\|)^t$ at the right-boundary $x = x_m$.

By applying Eq. (20) step-by-step we can obtain

$$\mathbf{X}^m = \mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)\mathbf{X}^0. \tag{24}$$

However, let us recall that each $\mathbf{G}_i, i = 1, \ldots, K$, is an element of the Lie-group $SO_o(2n,1)$, and by the closure property of the Lie-group, $\mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)$ is also a Lie-group denoted by $\mathbf{G}$. Hence, from Eq. (24) it follows that

$$\mathbf{X}^m = \mathbf{G}\mathbf{X}^0. \tag{25}$$

This is a one-step transformation from $\mathbf{X}^0$ to $\mathbf{X}^m$.

It should be stressed that the one-step Lie-group transformation property is usually not shared by other numerical methods, because those methods not necessarily belong to the Lie-group schemes. This important property has first pointed out by Liu [23] and used to solve the backward in time Burgers equation. After that Liu [22] has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended to estimate thermophysical properties of heat conductivity and heat capacity [24–26].

The remaining problem is how to calculate $\mathbf{G}$. While an exact solution of $\mathbf{G}$ is not available, we can calculate $\mathbf{G}$ through a numerical method by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie-group generated from such an $\mathbf{A} \in so(2n,1)$ by an exponential mapping is

$$\mathbf{G}(r) = \begin{bmatrix} \mathbf{I}_{2n} + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^t & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^t}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{26}$$

where

$$\hat{\mathbf{y}} = r\mathbf{y}^0 + (1-r)\mathbf{y}^m, \tag{27}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{x}, \hat{\mathbf{y}}), \tag{28}$$

$$a = \cosh\left(\frac{x_m\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right), b = \sinh\left(\frac{x_m\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right). \tag{29}$$

Here, we use the left-side $\mathbf{y}^0 = (\mathbf{T}(0), \mathbf{S}(0))$ and the right-side $\mathbf{y}^m = (\mathbf{T}(x_m), \mathbf{S}(x_m))$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $r \in (0,1)$ is a parameter to be determined and $\hat{x} = (1-r)x_m$. To stress its dependence on r we have denoted this $\mathbf{G}$ by $\mathbf{G}(r)$.

2.4. A Lie-group mapping between two points on the cone

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{y}}\|}, \tag{30}$$

such that Eqs. (26) and (29) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{2n} + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{31}$$

$$a = \cosh(x_m\|\mathbf{F}\|), b = \sin h(x_m\|\mathbf{F}\|). \tag{32}$$

From Eqs. (15), (25) and (31) it follows that

$$\mathbf{y}^m = \mathbf{y}^0 + \eta\mathbf{F}, \tag{33}$$

$$\|\mathbf{y}^m\| = a\|\mathbf{y}^0\| + b\frac{\mathbf{F}\cdot\mathbf{y}^0}{\|\mathbf{F}\|}, \tag{34}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{y}^0 + b\|\mathbf{y}^0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{35}$$

Substituting $\mathbf{F}$ in Eq. (33), written as

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}^m - \mathbf{y}^0), \tag{36}$$

into Eq. (34) and dividing both the sides by $\|\mathbf{y}^0\| > 0$, we obtain

$$\frac{\|\mathbf{y}^m\|}{\|\mathbf{y}^0\|} = a + b\frac{(\mathbf{y}^m - \mathbf{y}^0)\cdot\mathbf{y}^0}{\|\mathbf{y}^m - \mathbf{y}^0\|\|\mathbf{y}^0\|}, \tag{37}$$

where, after inserting Eq. (36) for $\mathbf{F}$ into Eq. (32), a and b are now written as

$$a = \cosh\left(\frac{x_m\|\mathbf{y}^m - \mathbf{y}^0\|}{\eta}\right), \quad b = \sin h\left(\frac{x_m\|\mathbf{y}^m - \mathbf{y}^0\|}{\eta}\right). \tag{38}$$

Let

$$\cos\ \theta := \frac{(\mathbf{y}^m - \mathbf{y}^0)\cdot\mathbf{y}^0}{\|\mathbf{y}^m - \mathbf{y}^0\|\|\mathbf{y}^0\|}, \tag{39}$$

$$x_y := x_m\|\mathbf{y}^m - \mathbf{y}^0\|, \tag{40}$$

where $0 \leqslant \theta \leqslant \pi$ is the intersection angle between vectors $\mathbf{y}^m - \mathbf{y}^0$ and $\mathbf{y}^0$, and thus from Eqs. (37) and (38) it follows that

$$\frac{\|\mathbf{y}^m\|}{\|\mathbf{y}^0\|} = \cosh\left(\frac{x_y}{\eta}\right) + \cos\ \theta \sin h\left(\frac{x_y}{\eta}\right). \tag{41}$$

Upon defining

$$Z := \exp\left(\frac{x_y}{\eta}\right), \tag{42}$$

from Eq. (41) we obtain a quadratic equation for Z:

$$(1+\cos\ \theta)Z^2 - \frac{2\|\mathbf{y}^m\|}{\|\mathbf{y}^0\|}Z + 1 - \cos\ \theta = 0. \tag{43}$$

On the other hand, by inserting Eq. (36) for $\mathbf{F}$ into Eq. (35) we obtain

$$\|\mathbf{y}^m - \mathbf{y}^0\|^2 = (a-1)(\mathbf{y}^m - \mathbf{y}^0)\cdot\mathbf{y}^0 + b\|\mathbf{y}^0\|\|\mathbf{y}^m - \mathbf{y}^0\|. \tag{44}$$

Dividing both sides by $\|\mathbf{y}^0\|\|\mathbf{y}^m - \mathbf{y}^0\|$ and using Eqs. (38), (39), (40) and (42) we obtain another quadratic equation for Z:

$$(1+\cos\ \theta)Z^2 - 2\left(\cos\ \theta + \frac{\|\mathbf{y}^m - \mathbf{y}^0\|}{\|\mathbf{y}^0\|}\right)Z + \cos\ \theta - 1 = 0. \tag{45}$$

From Eqs. (43) and (45), the solution of Z is found to be

$$Z = \frac{(\cos\ \theta - 1)\|\mathbf{y}^0\|}{\cos\ \theta\ \|\mathbf{y}^0\| + \|\mathbf{y}^m - \mathbf{y}^0\| - \|\mathbf{y}^m\|}. \tag{46}$$

From Eqs. (42) and (40) it follows that

$$\eta = \frac{x_m\|\mathbf{y}^m - \mathbf{y}^0\|}{\ln Z}. \tag{47}$$

Therefore, we come to an important result that between any two points $(\mathbf{y}^0, \|\mathbf{y}^0\|)$ and $(\mathbf{y}^m, \|\mathbf{y}^m\|)$ on the cone, there exists a Lie-group element $\mathbf{G} \in SO_o(2n,1)$ mapping $(\mathbf{y}^0, \|\mathbf{y}^0\|)$ onto $(\mathbf{y}^m, \|\mathbf{y}^m\|)$, which is given by

$$\begin{bmatrix} \mathbf{y}^m \\ \|\mathbf{y}^m\| \end{bmatrix} = \mathbf{G}\begin{bmatrix} \mathbf{y}^0 \\ \|\mathbf{y}^0\| \end{bmatrix}, \tag{48}$$

where $\mathbf{G}$ is given by the following equations:

$$\mathbf{G}(x_m) = \begin{bmatrix} \mathbf{I}_{2n} + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{49}$$

$$a = \cosh(x_m\|\mathbf{F}\|), b = \sinh(x_m\|\mathbf{F}\|), \tag{50}$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}^m - \mathbf{y}^0) = \frac{\ln Z}{x_m}\frac{\mathbf{y}^m - \mathbf{y}^0}{\|\mathbf{y}^m - \mathbf{y}^0\|}. \tag{51}$$

In view of Eqs. (46) and (39), it can be seen that $\mathbf{G}$ is fully determined by $\mathbf{y}^0$ and $\mathbf{y}^m$.

It should be stressed that the above $\mathbf{G}$ is different from the one in Eq. (26). In order to feature its property as a Lie-group mapping between the quantities spanned a whole length x_m we write it to be $\mathbf{G}(x_m)$. Conversely, $\mathbf{G}(r)$ is a function of r. However, these two Lie-group elements $\mathbf{G}(r)$ and $\mathbf{G}(x_m)$ are both indispensable in our development of the TSLGSM in the next section for the inverse problem of heat source identification.

3. Two-stage Lie-group shooting method

From Eqs. (7)–(12) it follows that

$$\mathbf{T}' = \mathbf{S}, \tag{52}$$

$$\mathbf{S}' = \mathbf{h}(x, \mathbf{T}), \tag{53}$$

$$\mathbf{T}(0) = \mathbf{T}^0, \mathbf{T}(x_m) = \mathbf{T}^m, \tag{54}$$

$$\mathbf{S}(0) = \mathbf{S}^0,\ \mathbf{S}(x_m) = \mathbf{S}^m, \tag{55}$$

where $\mathbf{T}^0$ and $\mathbf{T}^m$ are known from Eqs. (10) and (12), but $\mathbf{S}^0$ and $\mathbf{S}^m$ are two unknown vectors. Below we derive algebraic equations to solve them.

By using Eq. (14) for $\mathbf{y}$ we have

$$\mathbf{y}^0 = \begin{bmatrix} \mathbf{T}^0 \\ \mathbf{S}^0 \end{bmatrix}, \quad \mathbf{y}^m = \begin{bmatrix} \mathbf{T}^m \\ \mathbf{S}^m \end{bmatrix}, \tag{56}$$

and further inserting them into Eq. (36) yields

$$\mathbf{F} := \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} \mathbf{T}^m - \mathbf{T}^0 \\ \mathbf{S}^m - \mathbf{S}^0 \end{bmatrix}. \tag{57}$$

Comparing Eq. (57) with Eq. (30) and using Eqs. (14) and (56), we can obtain

$$\mathbf{T}^m = \mathbf{T}^0 + \frac{\eta_1}{\|\hat{\mathbf{y}}_1\|} \hat{\mathbf{S}}_1, \tag{58}$$

$$\mathbf{S}^m = \mathbf{S}^0 + \frac{\eta_1}{\|\hat{\mathbf{y}}_1\|} \hat{\mathbf{h}}_1, \tag{59}$$

where

$$\|\hat{\mathbf{y}}_1\| = \sqrt{\|\hat{\mathbf{T}}_1\|^2 + \|\hat{\mathbf{S}}_1\|^2} = \sqrt{\|r\mathbf{T}^0 + (1-r)\mathbf{T}^m\|^2 + \|r\mathbf{S}^0 + (1-r)\mathbf{S}^m\|^2}, \tag{60}$$

$$\hat{\mathbf{h}}_1 = \mathbf{h}((1-r)x_m, \hat{\mathbf{T}}_1), \tag{61}$$

$$\cos\theta_1 := \frac{(\mathbf{T}^m - \mathbf{T}^0)\cdot \mathbf{T}^0 + (\mathbf{S}^m - \mathbf{S}^0)\cdot \mathbf{S}^0}{\sqrt{\|\mathbf{T}^m - \mathbf{T}^0\|^2 + \|\mathbf{S}^m - \mathbf{S}^0\|^2}\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2}}, \tag{62}$$

$$Z_1 = \frac{(\cos\theta_1 - 1)\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2}}{\cos\theta_1\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2} + \sqrt{\|\mathbf{T}^m - \mathbf{T}^0\|^2 + \|\mathbf{S}^m - \mathbf{S}^0\|^2} - \sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2}}, \tag{63}$$

$$\eta_1 = \frac{x_m\sqrt{\|\mathbf{T}^m - \mathbf{T}^0\|^2 + \|\mathbf{S}^m - \mathbf{S}^0\|^2}}{\ln Z_1}. \tag{64}$$

For the use in later $\hat{\mathbf{h}}_1$ is written explicitly as

$$\hat{\mathbf{h}}_1 = \begin{bmatrix} \frac{\hat{T}_1^2 - \hat{T}_1^0}{2\Delta t} - H_1 \\ \vdots \\ \frac{\hat{T}_1^n - \hat{T}_1^{n-2}}{2\Delta t} - H_{n-1} \\ \frac{3\hat{T}_1^n - 4\hat{T}_1^{n-1} + \hat{T}_1^{n-2}}{2\Delta t} - H_n \end{bmatrix}, \tag{65}$$

where $\hat{T}_1^i = rF_0(t_i) + (1-r)F_m(t_i), i = 1, \ldots, n$, and $\hat{T}_1^0 = f((1-r)x_m)$. We must stress that $\hat{\mathbf{h}}_1$ is an unknown vector due to the appearence of unknown heat source H_i.

Similarly, we can obtain the Lie-group shooting equations in the interval of $x \in [x_m, \ell]$:

$$\mathbf{T}^\ell = \mathbf{T}^m + \frac{\eta_2}{\|\hat{\mathbf{y}}_2\|} \hat{\mathbf{S}}_2, \tag{66}$$

$$\mathbf{S}^\ell = \mathbf{S}^m + \frac{\eta_2}{\|\hat{\mathbf{y}}_2\|} \hat{\mathbf{h}}_2, \tag{67}$$

where

$$\|\hat{\mathbf{y}}_2\| = \sqrt{\|\hat{\mathbf{T}}_2\|^2 + \|\hat{\mathbf{S}}_2\|^2} = \sqrt{\|r\mathbf{T}^m + (1-r)\mathbf{T}^\ell\|^2 + \|r\mathbf{S}^m + (1-r)\mathbf{S}^\ell\|^2}, \tag{68}$$

$$\hat{\mathbf{h}}_2 = \mathbf{h}(rx_m + (1-r)\ell, \hat{\mathbf{T}}_2), \tag{69}$$

$$\cos\theta_2 := \frac{(\mathbf{T}^\ell - \mathbf{T}^m)\cdot \mathbf{T}^m + (\mathbf{S}^\ell - \mathbf{S}^m)\cdot \mathbf{S}^m}{\sqrt{\|\mathbf{T}^\ell - \mathbf{T}^m\|^2 + \|\mathbf{S}^\ell - \mathbf{S}^m\|^2}\sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2}}, \tag{70}$$

$$Z_2 = \frac{(\cos\theta_2 - 1)\sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2}}{\cos\theta_2\sqrt{\|\mathbf{T}^m\|^2 + \|\mathbf{S}^m\|^2} + \sqrt{\|\mathbf{T}^\ell - \mathbf{T}^m\|^2 + \|\mathbf{S}^\ell - \mathbf{S}^m\|^2} - \sqrt{\|\mathbf{T}^\ell\|^2 + \|\mathbf{S}^\ell\|^2}}, \tag{71}$$

$$\eta_2 = \frac{(\ell - x_m)\sqrt{\|\mathbf{T}^\ell - \mathbf{T}^m\|^2 + \|\mathbf{S}^\ell - \mathbf{S}^m\|^2}}{\ln Z_2}. \tag{72}$$

The above $\mathbf{S}^0$, $\mathbf{S}^m$, $\mathbf{S}^\ell$, $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$ are unknown vectors but the three vectors $\mathbf{T}^0$, $\mathbf{T}^m$, and $\mathbf{T}^\ell$ are known and given by

$$\mathbf{T}^0 = \begin{bmatrix} F_0(t_1) \\ \vdots \\ F_0(t_n) \end{bmatrix}, \quad \mathbf{T}^m := \begin{bmatrix} F_m(t_1) \\ \vdots \\ F_m(t_n) \end{bmatrix}, \quad \mathbf{T}^\ell := \begin{bmatrix} F_\ell(t_1) \\ \vdots \\ F_\ell(t_n) \end{bmatrix}. \tag{73}$$

We can evaluate these unknown vectors as follows. By using

$$\hat{\mathbf{S}}_1 = r\mathbf{S}^0 + (1-r)\mathbf{S}^m, \hat{\mathbf{S}}_2 = r\mathbf{S}^m + (1-r)\mathbf{S}^\ell, \tag{74}$$

from Eqs. (58), (59), (66) and (67) we can solve

$$\hat{\mathbf{h}}_1 = \frac{\|\hat{\mathbf{y}}_1\|}{\eta_1}(\mathbf{S}^m - \mathbf{S}^0), \tag{75}$$

$$\mathbf{S}^0 = \frac{\|\hat{\mathbf{y}}_1\|}{\eta_1}(\mathbf{T}^m - \mathbf{T}^0) - \frac{(1-r)\eta_1}{\|\hat{\mathbf{y}}_1\|}\hat{\mathbf{h}}_1, \tag{76}$$

$$\mathbf{S}^m = \frac{\|\hat{\mathbf{y}}_2\|}{\eta_2}(\mathbf{T}^\ell - \mathbf{T}^m) - \frac{(1-r)\eta_2}{\|\hat{\mathbf{y}}_2\|}\hat{\mathbf{h}}_2, \tag{77}$$

$$\mathbf{S}^\ell = \frac{\|\hat{\mathbf{y}}_2\|}{\eta_2}(\mathbf{T}^\ell - \mathbf{T}^m) + \frac{r\eta_2}{\|\hat{\mathbf{y}}_2\|}\hat{\mathbf{h}}_2, \tag{78}$$

where

$$\hat{\mathbf{h}}_2 = \begin{bmatrix} \frac{\hat{T}_2^2 - \hat{T}_2^0}{2\Delta t} - H_1 \\ \vdots \\ \frac{\hat{T}_2^n - \hat{T}_2^{n-2}}{2\Delta t} - H_{n-1} \\ \frac{3\hat{T}_2^n - 4\hat{T}_2^{n-1} + \hat{T}_2^{n-2}}{2\Delta t} - H_n \end{bmatrix} = \begin{bmatrix} \frac{(\hat{T}_2^2 - \hat{T}_2^0)(\hat{h}_1^1 + H_1)}{\hat{T}_1^2 - \hat{T}_1^0} - H_1 \\ \vdots \\ \frac{(\hat{T}_2^n - \hat{T}_2^{n-2})(\hat{h}_1^{n-1} + H_{n-1})}{\hat{T}_1^n - \hat{T}_1^{n-2}} - H_{n-1} \\ \frac{(3\hat{T}_2^n - 4\hat{T}_2^{n-1} + \hat{T}_2^{n-2})(\hat{h}_1^n + H_n)}{3\hat{T}_1^n - 4\hat{T}_1^{n-1} + \hat{T}_1^{n-2}} - H_n \end{bmatrix} \tag{79}$$

with $\hat{T}_2^i = rF_m(t_i) + (1-r)F_\ell(t_i), i = 1, \ldots, n$ and $\hat{T}_2^0 = f(rx_m + (1-r)\ell)$, can be obtained from $\hat{\mathbf{h}}_1$.

Because $\mathbf{T}^0$, $\mathbf{T}^m$ and $\mathbf{T}^\ell$ are all available, for a specified r, we can use Eqs. (76)–(79), starting from an initial guess, saying $(\mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^\ell) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, to generate a new $(\mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^\ell)$, until they converge according to a given stopping criterion:

$$\sqrt{\|\mathbf{S}_{i+1}^0 - \mathbf{S}_i^0\|^2 + \|\mathbf{S}_{i+1}^m - \mathbf{S}_i^m\|^2 + \|\mathbf{S}_{i+1}^\ell - \mathbf{S}_i^\ell\|^2} \leqslant \epsilon, \tag{80}$$

which means that the norm of the difference between the $i+1$-th and the i-th iterations of $(\mathbf{S}^0, \mathbf{S}^m, \mathbf{S}^\ell)$ is smaller than ϵ.

If the new $\hat{\mathbf{h}}_1$ is available, then by Eq. (65) we can calculate H_i by

$$H_i = \frac{\hat{F}(t_{i+1}) - \hat{F}(t_{i-1})}{2\Delta t} - \hat{h}_1^i, i = 1, \ldots, n-1, \tag{81}$$

$$H_n = \frac{3\hat{F}(t_n) - 4\hat{F}(t_{n-1}) + \hat{F}(t_{n-2})}{2\Delta t} - \hat{h}_1^n, \tag{82}$$

where $\hat{h}_1^i$ denotes the i-th component of $\hat{\mathbf{h}}_1$ and $\hat{F}(t_i) = rF_0(t_i) + (1-r)F_m(t_i)$ is known for the specified r.

Under the above new left-boundary condition $\mathbf{S}^0$ together with the known boundary condition of $\mathbf{T}^0$ and the new H_i, we can return to Eqs. (7)–(9) and integrate them to obtain $\mathbf{T}(x_m)$ and $\mathbf{S}(x_m)$. The above process can be done for all r in the interval of $r \in (0,1)$. Among these solutions we can pick up the best r, which leads to the smallest error of

$$\min_{r\in(0,1)} \sqrt{\|\mathbf{T}(x_m) - \mathbf{T}^m\|^2}, \tag{83}$$

such that the overspecified condition in Eq. (4) can be fulfilled as best as possible.

When the process terminates, by inserting the best r and $\hat{h}_1^i$ into Eqs. (81) and (82) we can estimate the time-dependent heat source $H(t)$ at the discretized times t_i. The above method will be called a two-stage Lie-group shooting method (TSLGSM).

4. Numerical examples

Now, we are ready to apply the TSLGSM on the estimations of $H(t)$ through the tests of numerical examples. We are concerned with the stability of TSLGSM by adding different levels of random noise on the measured data:

$$\hat{F}_m(t_i) = F_m(t_i) + sR(i), \tag{84}$$

where $F_m(t_i)$ is the exact data, s specifies the level of noise, and $R(i)$ are random numbers in $[-1,1]$.

4.1. Example 1

Let us first consider a simple inverse heat source problem with an exact solution of $H(t) = -6t$, where $T(x,t)$ is given by

$$T(x,t) = e^{-t}\sin x + 3tx^2 + \frac{1}{4}x^4. \tag{85}$$

The given data $F_0(t)$, $F_\ell(t)$, $f(x)$ and $F_m(t)$ can be computed from the exact solution. In this case we take $\ell = 0.1$ and without exception $x_m = \ell/2$. In addition we take $t_f = 1$, $\Delta t = 0.01$ and $\Delta x = 0.002$.

Before employing the numerical method of TSLGSM to calculate this example we use it to demonstrate how to pick up the best r as specified by Eq. (83). In the calculation we fixed the stopping criterion used in Eq. (80) to be $\epsilon = 10^{-3}$. We plot the error of mis-matching the target with respect to r in Fig. 1(a) in a finer range of $0.4 < r < 0.6$. It can be seen that there is a minimum point. Under this r the left-boundary condition and the calculated H_i derived from the TSLGSM provide the best match to the right-boundary condition at x_m. Then we can use the given $\mathbf{T}^0$ and the estimated $\mathbf{S}^0$ to calculate the whole temperature in the rod. In Fig. 1(b) we compare the exact H with the numerical one, of which the numerical error as shown in Fig. 1(c) is smaller than 10^{-2}.

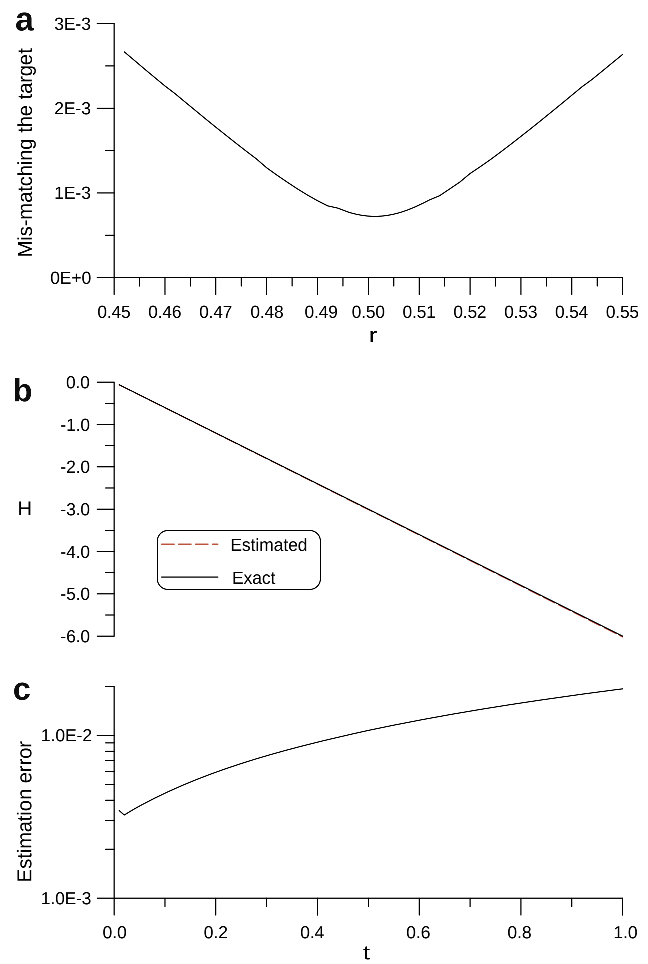

Fig. 1. For example 1: (a) plotting the error of mis-matching the target with respect to r in a finer interval, (b) comparing the numerical result with exact result, and (c) displaying the estimation error.

4.2. Example 2

In order to further explore the applicability of this TSLGSM we consider a slightly complex problem with $T(x,t)$ given by

$$T(x,t) = x^2 + 2t + \sin(2\pi t), \tag{86}$$

and $H(t)$ given by $H(t) = 2\pi\cos(2\pi t)$.

In Fig. 2(a) we compare the numerical solution of H with the exact one in the time interval of $t \in [0,1]$. These two curves are almost coincident, and the error is plotted in Fig. 2(b), which is smaller than 5×10^{-3}.

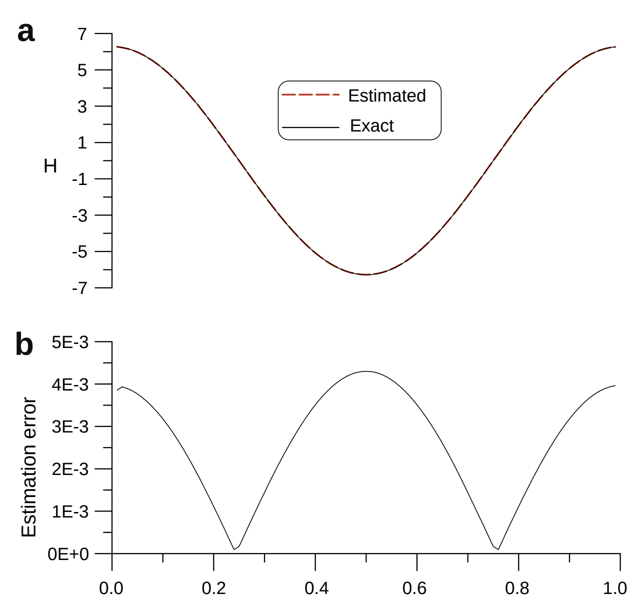

Fig. 2. For example 2: (a) comparing the numerical result with exact result and (b) displaying the estimation error.

4.3. Example 3

Let us consider the following example with $H(t)$ given by

$$H(t) = \begin{cases} 2 & t \in [0, 0.3), \\ 4 & t \in [0.3, 0.6), \\ 2 & t \in [0.6, 0.9), \\ 4 & t \in [0.9, 1.2), \\ 2 & t \in [1.2, 1.5), \\ 4 & t \in [1.5, 2]. \end{cases} \quad (87)$$

Here, we let $t_f = 2$. Subjecting to the boundary conditions:

$$T(0,t) = 2t, \quad T(\ell, t) = 2t + \ell, \quad (88)$$

and the initial condition

$$T(x,0) = x, \quad (89)$$

we can apply the TSLGSM to calculate this example. However, the temperature data at $x = x_m$ is calculated by using the Lie-group shooting method together with the fourth-order Runge–Kutta method (RK4) with $\Delta t = 0.02$ and $\Delta x = 0.01$. The data F_m are shown in Fig. 3(a).

Yan et al. [13] have calculated a similar example by using the method of fundamental solutions together with a Tikhonov regularization method. However, the numerical results as shown in Fig. 7 of the above cited paper are not so good. Because the heat conduction is an irreversible process in time, the solution gets smooth rapidly in time. The characteristic of solution in time may be smooth as shown in Fig. 4(a) for a certain case. It is hard to use the smooth data to recover the non-smooth heat source.

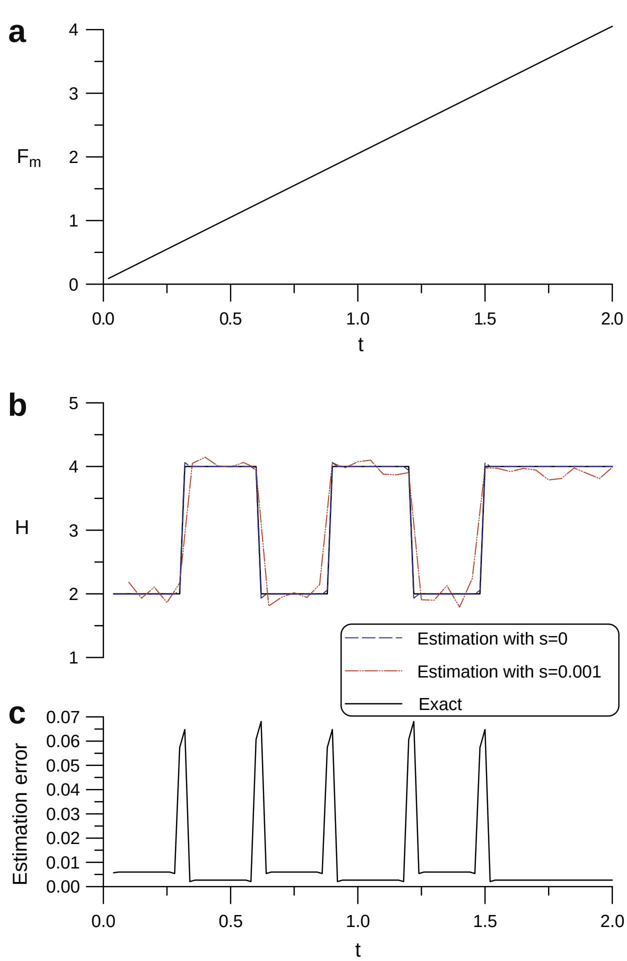

Fig. 3. For example 3: (a) plotting the data F_m, (b) comparing the numerical results under $s = 0$ and $s = 0.001$ with exact result, and (c) displaying the estimation error.

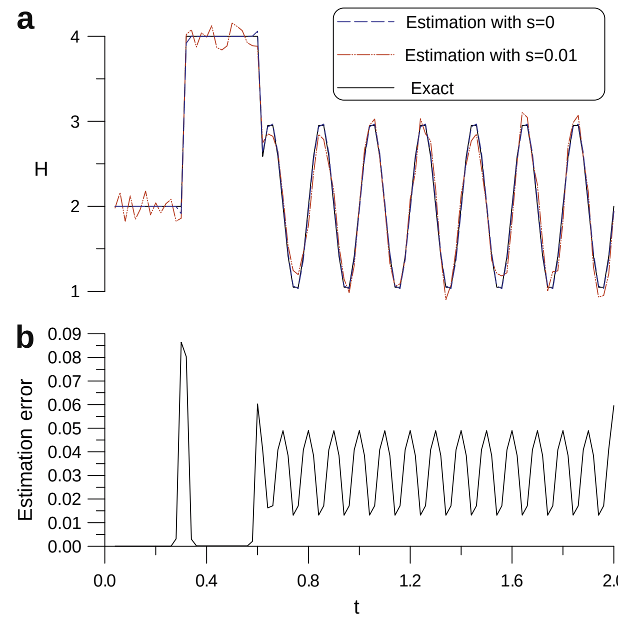

Fig. 4. For example 4: (a) comparing the numerical results under $s = 0$ and $s = 0.01$ with exact result and (b) displaying the estimation error.

In Fig. 3(b) we compare the numerical solution of H with the exact one given by Eq. (87) in the time interval of $t \in [0, 2]$. These two curves with dashed line and solid line are coincident well, and the error is plotted in Fig. 3(c), which is smaller than 0.07. In the case by adding a noise with a level $s = 0.001$ on the input data, we plot the numerical result in Fig. 3(b) by the dashed-dotted line. It can be seen that this method is robust against the noise.

4.4. Example 4

When the internal measuring point of temperature can be put as close as possible to the right-boundary we can calculate the temperature gradient at $x = \ell$ by

$$\frac{\partial T(\ell, t)}{\partial x} \approx \frac{T(\ell, t) - T(x_m, t)}{\ell - x_m}. \quad (90)$$

Thus the right-boundary condition of S^i is available. From Eqs. (58) and (59) with $x_m = \ell$ we can solve

$$\mathbf{S}^0 = \frac{\|\hat{\mathbf{y}}_1\|}{\eta_1}(\mathbf{T}^\ell - \mathbf{T}^0) - \frac{(1-r)\eta_1}{\|\hat{\mathbf{y}}_1\|}\hat{\mathbf{h}}_1, \quad (91)$$

$$\hat{\mathbf{h}}_1 = \frac{\|\hat{\mathbf{y}}_1\|}{\eta_1}(\mathbf{S}^\ell - \mathbf{S}^0). \quad (92)$$

By the same token we can develop a one-stage LGSM for this case. For a more detailed description of this method for estimating the time-dependent heat conductivity one may refer the paper by Liu [6].

In order to test this method on the estimation of discontinuous and oscillatory heat source, let us consider

$$H(t) = \begin{cases} 2 & t \in [0, 0.3), \\ 4 & t \in [0.3, 0.6), \\ 2 + \sin(10\pi t) & t \in [0.6, 2]. \end{cases} \quad (93)$$

Here, we let $t_f = 2$ and subject it to the same boundary conditions and initial condition as that in Eqs. (88) and (89). We can apply the one-stage LGSM to calculate this example.

The data $F_m(t_i)$ are obtained by applying a numerical method, for example the RK4, on the corresponding direct problem, supposing that $H(t)$ is known from Eq. (93). In this identification of $H(t)$ we have fixed $\Delta x = 0.01$ and $\Delta t = 0.01$ for the un-noised case, and $\Delta x = 0.01$ and $\Delta t = 0.05$ for the noised case. In Fig. 4(a) we compare the numerical solution of $H(t)$ with exact solution. The errors are rather small in the order of 10^{-2} as shown in Fig. 4(b). From this example one may appreciate the accuracy of the LGSM provided here even for identifying a highly discontinuous and oscillatory parameter $H(t)$ in the above. By adding a noise with $s = 0.01$ the result as shown in Fig. 4(a) by the dashed-dotted line reveals that it is more robust against the noise than the TSLGSM. In this case the accuracy is not so good as in the first two examples, whose reason is attributed to that this function of $H(t)$ is more difficult to estimate, and the input data $F_m(t_i)$ are not exact.

5. Conclusions

In order to estimate the time-dependent heat source under an extra measured temperature at an internal point, we have employed the TSLGSM to derive algebraic equations and solved them by iteration process. Numerical examples were worked out, which show that our TSLGSM is applicable even under a large noise on the measured data. Through this study, we can conclude that the new estimation method is accurate, effective and stable. Its numerical implementation is simple and the computational cost is low. When an internal measurement of temperature is near to the boundary, the one-stage LGSM is suggested, because the one-stage LGSM is less complex than the TSLGSM and its robustness is also good than the TSLGSM. According to these facts, this TSLGSM and one-stage LGSM can be used in practice as an accurate and effective mathematical tool to estimate the unknown time-dependent heat source.

Acknowledgement

Taiwan's National Science Council project NSC-97-2221-E-019-009-MY3 granted to the author is highly appreciated.

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